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A MODEL OF PROCEDURAL DECISION MAKING **IN THE PRESENCE OF RISK***

BY JUAN DUBRA AND EFE A. OK^1

Department of Economics, New York University, Universidad de Montevideo; Department of Economics, New York University

We introduce a procedural model of risky choice in which an individual is endowed with a core preference relation that may be highly incomplete. She can, however, derive further rankings of alternatives from her core preferences by means of a procedure based on the independence axiom. We find that the preferences that are generated from an initial set of rankings according to this procedure can be represented by means of a set of von Neumann-Morgenstern utility functions, thereby allowing for incompleteness of preference relations. The proposed theory also yields new characterizations of the stochastic dominance orderings.

1. INTRODUCTION

While the expected utility theory is still very widely used, a number of interesting decision theories that generalize the expected utility paradigm have been recently developed. For the large part, these theories axiomatically devise preference functionals that do not necessarily take the expected utility form, but that nevertheless lead to the ranking of *all* lotteries by the individuals. Surprisingly, the alternative path of generalization in which the expected utility form of the preference functionals is retained, but a degree of incompleteness is allowed, has not been pursued, the seminal contributions of Aumann (1962) and Bewley (1986) being two major exceptions. This is somewhat unfortunate, because incomplete preferences are not only descriptively plausible, but they also conveniently leave room for introducing to the theory empirically observed behaviors like status quo bias and loss aversion (Bewley, 1986; Mandler, 1999).²

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 2 There are several conceptual arguments that have been put forth in the literature in favor of incomplete preferences. For example, it has been justly argued that allowing for such preferences in multicriteria decision problems is in the nature of things. Since Aumann (1962) and Bewley (1986), inter alia, go through these arguments in considerable detail, we shall not reproduce them here. As a historical note, however, we would like to mention that the overly restrictive nature of the completeness

Our interest in incomplete preferences stems from the fact that such preferences allow one to introduce "procedural" elements into the decision-making analysis. Procedural rationality maintains that individuals often use some form of reasoning, or a deduction process, when "solving" the choice problems they face. While it is difficult to argue against this alternative notion of rationality, especially from a descriptive point of view, it is not clear how to build a decision theory around it. The difficulty is that such a theory requires one to discover and formalize the decision procedures that individuals are presumably using. Nevertheless, there are several avenues to pursue in this regard, and as a matter of fact, some promising attempts of doing this are already present in the literature (cf. Rubinstein, 1988, 1998; Gilboa and Schmeidler, 1995). These studies, however, do not really aim to provide a general method of attack toward a theory of procedural decision making. By contrast, our aim in this article is precisely to develop such a method in the classical von Neumann-Morgenstern framework. Our approach is in part axiomatic, and takes as a basis the tendency of decision makers to use some rules/procedures in going from simple decisions to complex ones.

We propose to think of the decision maker as having a core preference relation over certain alternatives that consist of the primitive judgements of the individual. This core preference relation is presumably highly incomplete, for it corresponds to the rankings of the lotteries that are, in a sense, trivial for the individual (like the rankings of degenerate lotteries with monetary outcomes). The next ingredient of the model is a procedure that lets the agent deduce additional rankings. In the present article, we use a particular procedure for this, one that is based directly on the standard independence axiom. That is, we posit that an agent who initially ranks a lottery *p* higher than *q* deduces the superiority of a given mixing of *p* with another lottery over the same mixing of *q*. (In fact, this is how the independence axiom is usually motivated.) In this way, one extends the core judgements of the agent to a larger set, and then uses this extended set to obtain new rankings via applying the same procedure, and so on. The final outcome of this process is, then, viewed as the preference relation of the agent.³

It is critical that the "procedure" that we use in this model is based directly on the independence axiom. Of course, we do not have to confine attention to such a procedure in the sense that the model we suggest here remains meaningful with all sorts of other procedures. Having said this, however, we believe that using a procedure that is based on the independence axiom is a natural place to start. At the very least, this allows us to readily compare the conclusions of the present model with those of the standard one, and thus helps us better understand the contribution of the procedural rationality approach. What is more, as we shall show, such a procedure leads us to quite pleasing results that are readily applicable in economic models.

axiom was acknowledged by von Neumann and Morgenstern (1944, p. 630) who said about it that "it is very dubious, whether the idealization of reality which treats this postulate as a valid one, is appropriate or even convenient."

³ Although his main objective was somewhat different than ours, this particular formulation of procedural decision making was actually elaborated upon by Aumann (1962). It is, thus, quite surprising that the idea has not been pursued in the literature so far.

1055

The present model is explained, both formally and informally, in Section 2. It will suffice here to say that the preference relations that are generated by the procedure just described are typically (but not necessarily) incomplete, but they still allow for a useful expected utility representation. In the course of the following analysis, we shall show that such a preference relation, call it \succeq , has the following structure:

 $p \succeq q$ if and only if $E_p(u) \ge E_q(u)$ for all $u \in \mathcal{U}$

where \mathcal{U} is a set of cardinal utility functions, and $E_r(u)$ denotes the expected value of u with respect to the lottery r = p, q. Thus, such procedurally generated preference relations enjoy a vector-valued expected utility representation.

By varying the set of initial rankings of the individual (her core preference relation), we shall obtain in Sections 3 and 5 a number of such representation results; these results, along with a uniqueness theorem presented in Section 4, constitute the main theoretical contribution of the present work. They may be thought of as interesting for a variety of reasons. First, they introduce a conceptualization of procedural rationality that leads to a parsimonious departure from the expected utility paradigm. Second, they build a bridge between procedural rationality and multiobjective planning, which was informally suggested by Simon (1955). In this sense, our representation results are quite operational. Third, as explained in Section 5, these results link the present approach to the theory of stochastic orders. The first- and second-order stochastic dominance orderings are procedurally generated in our sense, and what is more, it is also easy to identify the core preference relations that generate these orderings. This allows us to obtain new characterizations of these celebrated preorders, and to give an immediate applicability to the procedural rationality theory that we propose here. To illustrate, two elementary applications are provided in Section 6, which is followed by sections outlining our final comments on future research and the proofs of the main results.

2. A MODEL OF PROCEDURAL DECISION MAKING

2.1. Procedural Rationality: An Informal Discussion. In a series of articles, Herbert Simon has stressed the distinction between what he referred to as *sub-stantial* rationality and *procedural* rationality. "Behavior is substantively rational when it is appropriate to the achievement of given goals within the limits imposed by given conditions and constraints. ... Behavior is procedurally rational when it is the outcome of appropriate deliberation" (Simon, 1976, pp. 425–26). It would be fair to say that standard economic analysis is based almost exclusively on the postulate of substantial rationality. However, mounting experimental evidence and growing literature on cognitive psychology have recently led some economic theorists to take the procedural aspects of decision making more seriously.⁴

Unfortunately, it is not at all obvious how to formalize procedural rationality in an operational way. The existing literature focuses mostly on a particular dimension of this concept, namely, the tendency of decision makers to simplify

⁴ See Chapters 1 and 2 of Rubinstein (1998) for an introduction to these matters.

matters by basing their decisions on some sort of a similarity relation. It is argued that people often evaluate two distinct alternatives ignoring the attributes of these alternatives that are "similar" to each other, or that they make use of past decisions that were made in "similar" circumstances. A formalization of the first idea was provided by Rubinstein (1988), whereas the latter gave rise to the so-called case-based decision theory that was first introduced by Gilboa and Schmeidler (1995).

Another major dimension of procedural rationality is, on the other hand, the tendency of the decision makers to use some rules in going from simple decisions to complex ones. In many decision-theoretic scenarios, it is possible to identify unambiguous problems that an overwhelming majority will find easy to solve. Offer a person a new insurance policy that is superior to the one she already has in every conceivable dimension, she will take it; ask a person to choose between two degenerate lotteries, one giving \$1 and the other \$100, she will know what to choose. Obviously, it is possible to give numerous trivial examples to this effect. But few of the interesting decision problems in life are of this trivial nature. When faced with a more complex problem, what does a decision maker do? An intuitive (and indeed procedural) argument would be that she tries to somehow "break down" the problem into smaller, easier problems that she knows how to solve. If she can really do this (that is, the problem can indeed be decomposed into "easy subproblems"), then she can comfortably make her decision. So, one way of thinking about a "procedurally rational" individual is to view her as knowing how to rank certain alternatives with ease, and then using a certain procedure to obtain solutions to new choice problems that she faces by using her primitive ranking. This view seems integral to some of the discussions of procedural rationality in the literature on behavioral decision making and cognitive psychology, but it is, to the best of our knowledge, not investigated formally yet. Our aim in this article is, then, to sketch a theory of decision making that corresponds to this point of view, and that retains some of the basic features of the classical expected utility theory.

Before moving on to the formal development, however, let us note that although procedural decision making is naturally motivated from a descriptive perspective, it is also useful for normative applications. For, instead of devising a complete plan of action for every possible contingency, it is often easier to come up with appealing rules that would help one go from simple decisions to complex ones. Thus, examining the implications of such procedures is of normative interest, and much of what follows has clear normative implications. In this sense, a procedural decision making model is not different from the expected utility model; either approach has its own pros and cons with respect to both normative and descriptive points of view.

2.2. Preliminary Notation and Definitions. Throughout this article, Z will stand for the nonempty set of prizes (outcomes). Although we shall impose further structure on it later on, for the general theory, we assume only that Z is a compact metric space. The set of all continuous real functions on Z endowed with the sup norm is denoted by C(Z). Strictly speaking, by a *utility function* in this article we mean any map $u : Z \rightarrow \mathbf{R}$. However, all utility functions that we shall

consider here will be continuous, and thus one can identify C(Z) with the space of all utility functions.

We denote the set of all Borel probability measures on Z by $\mathbb{P}(Z)$ and, as usual, view it as the space of all lotteries. (When we need to impose a topology on this space, we shall always use the topology of weak convergence.) The generic members of $\mathbb{P}(Z)$ are denoted p, q, r, etc. The expected value of a probability measure p is denoted E(p). The support of p, denoted S(p), is defined as the smallest closed set of full measure in Z. A probability measure is said to be *simple* if it has finite support, and *degenerate* if it has singleton support. We denote the set of all simple probability measures on Z by $\mathbb{P}_s(Z)$, and the degenerate lottery that puts the entire mass on the outcome z by δ_z .

By a preference relation \succeq in this article we mean a reflexive and transitive binary relation (i.e., a preorder) on $\mathbb{P}(Z)$. The strict part of \succeq is denoted by \succ and the indifference part by \sim . Although the interpretation of \succeq is identical to that in the expected utility theory, we do not assume here that a preference relation is necessarily complete. Upper and lower semicontinuity properties of \succeq are defined as usual.

2.3. Procedural Rationality: A Formal Discussion. We are interested in this article in those preference relations on $\mathbb{P}(Z)$ that are "derived" from a set of primitive comparisons of lotteries by means of a certain procedure. The beginning point of our procedural decision-making theory is, then, a preorder R on $\mathbb{P}(Z)$ that consists of the "primitive" judgements of the individual. Of course, R may consist of both indifference and strict preference rankings. Consequently, we decompose R into two parts as follows:

$$I_R := \{(p,q) \in R : (q,p) \in R\}$$
 and $P_R := R \setminus I_R$

If *R* is a complete ranking of lotteries, then the individual is inherently capable of ranking any two lotteries without using any simplifying procedure, and hence we do not need to go any further. The theory becomes more interesting, however, if *R* is incomplete; recall that we view the statement $(p, q) \in R$ as meaning that the individual has no doubt in her mind that *p* is at least as good as *q*. In this case, we may capture the idea that the actual preference relation \succeq is derived from *R* via some process by viewing it as a *proper extension* of *R*, that is, as a preorder on $\mathbb{P}(Z)$ such that $R \subseteq \succeq$ and

(1)
$$(p,q) \in P_R$$
 implies $p \succ q$

for all $p, q \in \mathbb{P}(Z)$. (We shall refer to the requirement (1) as "properness" in what follows.⁵) This suggests further that we may be able to model the process with which \succeq is derived from *R* by requiring that \succeq satisfies certain properties.

⁵ The properness requirement assures that if the agent ranks a lottery strictly better than another, then the procedure she uses will never lead her to view these alternatives indifferently. Given the interpretation of R, this is an unexceptionable requirement.

To be precise, suppose that $R = \{(p, q)\}$, which means that the only sure-fire judgement of the individual is the unambiguous superiority of p over q. Now suppose that this individual actually "thinks" in accordance with the standard independence axiom. Then, so the argument goes, she could use the independence property as a "procedure" to infer further rankings of lotteries from her only primitive ranking. Thus, for any $r \in \mathbb{P}(Z)$ and $\alpha \in (0, 1)$, such an individual could "break down" the problem of choosing between $\alpha p + (1 - \alpha)r$ and $\alpha q + (1 - \alpha)r$ to the problem of ranking p and q that she knows how to deal with. (Note that all we do here is to use the usual motivation for the independence axiom in an explicit way.) The totality of such inferences then defines the agent's preference relation that is derived from her primitive judgement(s) by using the procedure of (successively) applying the independence axiom. In this particularly simple case in which |R| = 1, the resulting preference relation is simply $\{(\alpha p + (1 - \alpha)r, \alpha q + (1 - \alpha)r), \alpha q \in [0, 1]\}$.

This particular way of thinking about procedural rationality makes it rather easy to formalize the approach in a workable format. To do this in the case of the particular procedure outlined above, let us state formally the independence axiom.

DEFINITION. Let \succeq be any preorder defined on $\mathbb{P}(Z)$. We say that \succeq *satisfies the independence axiom* if

$$p \succeq q$$
 iff $\alpha p + (1 - \alpha)r \succeq \alpha q + (1 - \alpha)r$

holds for all $p, q, r \in \mathbb{P}(Z)$ and $\alpha \in (0, 1)$.

The discussion outlined above, then, corresponds formally to saying this: The preference relation of the individual whose initial judgements are contained in the preorder *R*, and who uses the above described procedure is simply *the smallest proper extension of R that satisfies the independence axiom*, which we denote by $\succeq^{R.6}$ So we may think of the preference relation \succeq of an individual as generated from *R* by a procedure based on the independence property whenever $\succeq = \succeq^{R}$.

In summary, the procedural decision-making model that we propose here is made up of two components. The first is a (presumably highly incomplete) preorder *R* that consists of the primitive rankings of the individual. The second is a rationality property the successive application of which is viewed as the procedure the decision maker uses to obtain new rankings from those contained in *R*. Clearly, the nature of the induced preference relation depends on both *R* and the chosen rationality property. Our aim in this article is to demonstrate the promise of this model by examining those preferences that are derived from various choices of *R* by taking as the rationality property the standard independence axiom. Formally put, then, our objective is to identify the structure of \gtrsim^R for various choices of *R*.

Before proceeding, however, several caveats about the present approach are in order. First of all, as noted in Section 1, although we regard the procedures based on the independence axiom as naturally motivated, given the key role this

⁶ That is, $\sum_{n=1}^{R}$ is the intersection of all proper extensions of R, provided that $\sum_{n=1}^{R}$ is well defined.

property plays in decision theory at large, it is by no means clear that this is the optimal way of making use of the procedural model. In fact, we view procedural rationality as a systematic path toward a satisfactory theory of bounded rationality, and thus recognize that the independence axiom is likely to prove limiting for the workings of the theory. This axiom is consistently refuted in the experiments, and many weaker alternatives of it are thus suggested in the literature.⁷ However, it is still the key axiom of risk theory, and provides one with a good and tractable starting point. Moreover, by limiting ourselves to a procedure that is based on the independence axiom, we achieve a parsimonious departure from the expected utility paradigm. This, in turn, makes comparing the conclusions of the present model to those of the standard one an easy task.

Secondly, it should be clear that there is no guarantee that the present approach will yield a complete preference relation for a procedurally rational agent, thereby requiring us to work with incomplete orderings. Such preference relations are motivated and discussed extensively in the literature, and thus we will not elaborate on the issue of incompleteness here.⁸ Moreover, we are mainly interested here in obtaining representation theorems for procedurally derived preference relations in a sense that will become clear shortly. In particular, we shall not discuss here matters related to actual choices of the individuals, which would eventually require us to append to the present model certain conventions regarding the decision making (such as status quo bias, case-based decision making, expert consulting, etc.). Although extremely interesting, these issues lie outside our present scope, and will be taken up in future work.⁹

The third caveat is more specific to the subsequent development, and concerns the existence of \succeq^R . To see the basic nature of the difficulty, let $Z := \{a, b\}$ and $R := \{(\delta_a, \delta_b), (\delta_b, \delta_a), (p, q)\}$ where p and q are any two nondegenerate lotteries on Z. In other words, the individual's primitive judgement is such that she is indifferent to the degenerate lotteries δ_a and δ_b (so $(\delta_a, \delta_b) \in I_R$), whereas she strictly prefers p to q (so $(p, q) \in P_R$). It is easy to see here (by repeated application of the independence axiom) that any extension of R that satisfies the independence axiom must render all alternatives indifferent. Consequently, no such extension is proper, and thus \succeq^R does not exist.

 $^7\,{\rm Two}$ well-known surveys that discuss these matters at length are Machina (1987) and Camerer and Weber (1992).

⁸ See, in particular, Richter (1966), Peleg (1970), Mandler (1999), and Ok (2002) on the theory of incomplete preferences in the context of certainty. More related to the present work are, however, the important papers by Aumann (1962) and Bewley (1986), who discuss incomplete preferences in the context of choice under uncertainty. In particular, Aumann (1962) points to the link between the notions of incomplete preferences and procedural decision making in a way that parallels our approach.

⁹ We should, however, mention that there is certainly an interesting connection between the present approach based on incomplete preference relations and that of *satisficing behavior* that is powerfully advocated by Simon. As Sen (1990, p. 203) puts it, "... it can be argued that satisficing behavior really is maximization according to an effectively incomplete relation, such that the states satisfying the target level of achievement are all put in a non-comparable class as far as choice behavior is concerned. Maximization can indeed be defined in terms of such incomplete relations, ... and if it is seen in these terms, the gap between satisficing and maximizing may be, at least formally, reduced." This issue is further explored in Sen (1997).

This simple example shows us that one has to be careful when choosing the primitive relation R. Fortunately, however, it is relatively easy to determine a simple set of necessary and sufficient conditions on R that would guarantee the existence of \gtrsim^R (see Lemma 2). Roughly speaking, it is necessary and sufficient for R not to violate the independence axiom. The potential nonexistence of \gtrsim^R will thus not cause serious difficulties in the subsequent analysis.

3. REPRESENTATION OF FINITELY GENERATED PREFERENCES

Choosing a particular primitive ordering *R* obviously requires us to specify the decision-theoretic environment in more detail. At the level of generality that the von Neumann–Morgenstern theory is developed, there are not many interesting options to choose from. An extremely conservative approach might determine R as $\{(p, p)\}_{p \in \mathbb{P}(Z)}$, which means that the agent is initially fully indecisive, being unable to rank any two distinct lotteries. In this case, we obviously cannot reach interesting conclusions since $R = \succeq^R$. At the other end of the spectrum, one may postulate that *R* satisfies the von Neumann–Morgenstern axioms, in which case $R = \succeq^R$ holds again. The interesting scenario is clearly the one in which the agent has only a few rankings in mind, and the rest of her decisions are obtained by means of the procedure she uses. At this level of generality, the natural way of modeling "a few rankings" is by supposing that *R* is finite, and this is precisely what we shall do in this section.¹⁰

DEFINITION. A preorder \succeq on $\mathbb{P}(Z)$ is said to be generated by a finite procedure if there exists a finite set R such that $\succeq = \succeq^R$.

Given the discussion of the previous section, the interpretation of a preference relation that is generated by a finite procedure should be straightforward. We should perhaps add that the finiteness of R also helps one deduce the agent's primitive rankings by choice experiments, and then construct her entire preference relation by using successive applications of the independence axiom. This is, of course, nothing but a standard revealed preference argument that is particularly tractable here, thanks to the finiteness of R.^{11,12}

¹⁰ One should really view the finiteness of *R* here in excess of the trivial rankings $\{(p, p)\}_{p \in \mathbb{P}(Z)}$ that the individual surely possesses. These rankings will emerge in \succeq^R precisely because we assume this relation to be reflexive, and hence not including them in *R* is without loss of generality.

¹¹ Suppose that an experimenter would like to pin down the preference relation of an agent to deal with a particular problem, and for this reason, she conducts several experiments. The result will be finitely many rankings by the agent; call the totality of these *R*. With a little luck, *R* does not violate the independence axiom, and thus the experimenter may feel somewhat comfortable that the preference relation of the agent is of the von Neuman–Morgenstern type. But what is it exactly? It is impossible to answer this question in general with these givens, but this much we can say: The agent's preference relation is a proper extension of \gtrsim^{R} . Thus, it is of interest to study the structure of \gtrsim^{R} ; it may even be the case that \gtrsim^{R} is all the experimenter needs to solve her original problem.

¹² One may take issue with the term "finite procedure" that we use here, since it seems at first like the finiteness applies only to the cardinality of *R*. But this is really not the case; it is easy to verify that, if $p \gtrsim_{R}^{R} q$, then one can deduce this by applying the independence axiom at most |R| times.

As noted in Section 2.1, finitely generated preferences are also of interest from a normative point of view. From this angle, the independence axiom commands extensive support, and hence, loosely speaking, a preference relation that is generated by a finite procedure approximates the optimal decision making of a planner who is given a finite list of instructions.

Given these motivations, we shall inquire into the formal nature of finitely generated preference relations in the rest of this section.

3.1. *A General Representation Theorem.* Our main theorem concerns the representation of preference relations that are generated by a finite procedure.

THEOREM 1. If \succeq is a preference relation on $\mathbb{P}(Z)$ that is generated by a finite procedure, then there exists a convex set \mathcal{U} of continuous utility functions on Z such that

(2)
$$p \succeq q$$
 if and only if $\int_Z u \, dp \ge \int_Z u \, dq$ for all $u \in \mathcal{U}$

for each $p, q \in \mathbb{P}(Z)$.

Theorem 1 makes transparent the direction in which we depart from the classical paradigm. The extensive usage of the independence axiom (but not continuity) lets us obtain a theory very similar to the expected utility theory, albeit with one exception. In the present procedural setup, we find a number of utility functions as opposed to a single von Neumann–Morgenstern utility function. That is, a finitely generated preference relation ranks p above q iff the expected utility of p is higher than that of q with respect to a given *set* of utility functions. Put this way, such preference relations are reminiscent of stochastic orders (see Section 4). Moreover, they portray a dual nature relative to the recent decision-making approaches that are based upon nonunique priors (see Gilboa and Schmeidler, 1989; and especially Bewley, 1986). Although these approaches arrive axiomatically at a *set of probability measures*, and thus obtain several linear preference functionals (one for each subjective probability measure), our procedural approach yields a *set of utilities* and thus several linear preference functionals (one for each utility function).¹³

There is then a sense in which Theorem 1 is a vector-valued analogue of the classical expected utility theorem. (See also Theorem 2 below.) This is interesting in its own right, for it ties the procedural rationality model to the theory of multicriteria decision making that studies the problem of simultaneous maximization of several objective functions. After all, by Theorem 1, we may view a procedurally rational individual (to the extent that procedural rationality is captured by finitely generated preference relations) as a person who sees several dimensions

¹³ It is certainly an interesting question if one can also obtain such a multiutility representation result in an axiomatic way (for instance, by replacing the phrase "is generated by a finite procedure" with "satisfies the independence and continuity axiom." In the case of compact Z, this question was settled recently in the affirmative by Dubra et al. (2001). The basic problem is still open, however, in the case where Z is an arbitrary Polish space.

in a choice problem, and prefers an alternative over another when, and only when, this alternative performs better in all of these dimensions.

Before we move on to investigate some variants of Theorem 1, we should note that this result is indeed anticipated by the seminal Aumann (1962).¹⁴ However, Aumann's main objective in his related analysis was to understand under what conditions a given incomplete preference relation can be *extended* to a complete relation that admits an expected utility representation. Unfortunately, as Aumann himself has noted, this approach fails to provide positive results when the prize space is not finite. Although our approach is not axiomatic, Theorem 1 shows that the present procedural decision-making model yields an expected multiutility representation of an incomplete preference relation for very general prize spaces.

REMARK 1. When Z is infinite, there is a crucial difference between properly extending an incomplete preference relation and representing it by means of a set of utilities. To drive this point home, set Z := [0, 1], and let us agree to say that a preference relation \succeq on $\mathbb{P}(Z)$ admits an *Aumann utility* if there exists a function $u \in$ \mathbb{R}^{Z} such that $p \succ (\sim \text{resp.}) q$ implies $\int_{Z} u \, dp > (= \text{resp.}) \int_{Z} u \, dq$. This relation has a *multiutility representation* if there exists a set \mathcal{U} in \mathbb{R}^{Z} such that $p \succeq q$ iff $\int_{Z} u \, dp \ge \int_{Z} u \, dq$ for all $u \in \mathcal{U}$. It turns out that \succeq may not admit an Aumann utility even if it has a multiutility representation.

Define \succeq on $\mathbb{P}(Z)$ by

$$p \succeq q$$
 if and only if $\int_{Z} \mathbf{1}_{\{z\}} dp \ge \int_{Z} \mathbf{1}_{\{z\}} dq$ for all $0 \le z \le 1$

(that is, $p \succeq q$ iff $p(z) \ge q(z)$ for every atom of q). By definition, \succeq has a (Riemann integrable) multiutility representation. Yet \succeq does not have an Aumann utility. To see this, let $u \in \mathbf{R}^Z$ be an Aumann utility for \succeq that must be Borel measurable by definition. Now let ℓ denote the Lebesgue measure on [0, 1]. Since $\delta_1 > \ell$, the integral $\int_Z u \, d\ell$ must exist and satisfy $\infty > u(1) > \int_Z u \, d\ell$. It is also impossible that $\int_Z u \, d\ell = -\infty$, because this would contradict the fact that $\frac{2}{3}\delta_z + \frac{1}{3}\ell > \frac{1}{3}\delta_z + \frac{2}{3}\ell$. Therefore, u must be Lebesgue integrable, so $\alpha := \int_Z u \, d\ell$ is a real number. Moreover, for any $z \in Z$, we have $\delta_z > \frac{1}{2}\delta_z + \frac{1}{2}\ell$, and this implies that $u(z) > \alpha$ for all $z \in Z$. But of course, $\int_Z (u - \alpha) \, d\ell = 0$, which, given that $u - \alpha$ is a nonnegative random variable, may hold iff $u(z) = \alpha$ almost everywhere on Z, a contradiction.

3.2. Finite Prize Spaces. We have too little structure at present to be able to deal with the converse of Theorem 1. That is, Theorem 1 does not say if a preference relation defined by (2) is generated by a finite procedure or not. But in the case where Z is finite, this actually turns out to be true. What is more, in this

¹⁴ Under the assumption of finiteness of Z, Aumann (1962, p. 448) states that "... under fairly wide circumstances (but not always) we *can* say the following: Given the set of *all* utility functions, we can find the preference order. For example, this is true if we know a priori that the preference order is 'finitely generated' i.e., consists of a finite number of 'basic' preference statements that follow from these 'basic' ones by application of the axioms."

case we can also make sure that one uses only finitely many utilities in representing a finitely generated preference relation.

THEOREM 2. Let Z be a finite set. A preference relation \succeq on $\mathbb{P}(Z)$ is generated by a finite procedure if, and only if, there exists a positive integer n and a mapping $\mathbf{u}: Z \to \mathbf{R}^n$ such that

(3)
$$p \succeq q$$
 if and only if $\sum_{z \in Z} p(z)\mathbf{u}(z) \ge \sum_{z \in Z} q(z)\mathbf{u}(z)$ for all $p, q \in \mathbb{P}(Z)$

In the case of finite prize spaces, then, we are able to fully characterize the preference relations that are generated by a finite procedure in a way analogous to the expected utility theorem. In fact, it is easy to see that n = 1 can be chosen in Theorem 2, if \succeq is known to be complete, which should clarify the link between this result and the classical expected utility theorem. (Due to the straightforward summability problems, the same cannot be said in the general case.)

Being closely related to the problem of obtaining an expected multiutility representation for an incomplete preference relation, the "only if" part of Theorem 2 relates closely to Theorem D of Aumann (1962). However, it should be noted that Theorem 2 delivers more than Aumann's related result, because it guarantees the existence of *finitely* many utility functions that allow one to recover the original preference relation \gtrsim . Moreover, perhaps the more novel part of Theorem 2 is its "if" part. In particular, this observation helps one find interesting preorders that are generated by a finite procedure.

EXAMPLE 1. Let $Z = \{1, ..., d\}, d \in \mathbb{N}$ and let \geq_{FSD} stand for the first-order stochastic dominance ordering on $\mathbb{P}(Z)$, that is,

$$p \ge_{\text{FSD}} q$$
 iff $\sum_{z=1}^{\ell} p(z) \le \sum_{z=1}^{\ell} q(z)$ $\ell = 1, \dots, d-1$

(Here we interpret *d* as the best prize, d - 1 as the second best prize, and so on.) We claim that \geq_{FSD} is generated by a finite procedure. To see this, define, for each $j = 1, \dots, d - 1$,

 $u_i(z) := -1$ for each $z \le j$ and $u_i(z) := 0$ for each z > j

so that $\sum_{z \in Z} u_j(z) p(z) = -\sum_{z=1}^j p(z)$ for any $p \in \mathbb{P}(Z)$. Therefore,

$$p \ge_{\text{FSD}} q$$
 iff $\sum_{z \in Z} p(z) \mathbf{u}(z) \ge \sum_{z \in Z} q(z) \mathbf{u}(z)$ for all $p, q \in \mathbb{P}(Z)$

where $\mathbf{u} = (u_1, \ldots, u_{d-1})$. By Theorem 2, \geq_{FSD} must be generated by a finite procedure. A constructive proof of this claim is provided by verifying that $\geq_{\text{FSD}} = \sum^{R} \text{ with } R = \{(\delta_d, \delta_{d-1}), \ldots, (\delta_2, \delta_1)\}.$

We observe here that a first-order stochastic dominance maximizer can be thought of as a procedurally rational agent in a well-defined sense. This, in turn, shows again that the present theory is closely connected with the expected utility theory (which is, of course, due to the central role played by the independence axiom). We shall return to the connection between our approach and the stochastic dominance theory in Section 4.

In passing, we note that there is no upper bound on the number of utility functions needed to represent a finitely generated preference relation, when the prize space has at least four elements. We demonstrate this next by means of an example.

EXAMPLE 2. Let v be a regular polygon with n faces in \mathbb{R}^2 , and let $C \subset \mathbb{R}^3$ be the cone generated by the polygon $\{(x_1, x_2, 1) : x \in v\}$. To each face i of the cone C we may associate a vector $u_i \in \mathbb{R}^3$ that is orthogonal to that face, and such that $x \in C$ iff $u_i x \ge 0$ for each i. Let A be the 4×4 orthogonal matrix that maps $\mathbb{P}(Z) - (1, 0, 0, 0)$ to the plane $\mathbb{R}^3 \times \{0\}$ in \mathbb{R}^4 . Let $Z = \{1, 2, 3, 4\}$ and define the preference relation on $\mathbb{P}(Z)$ by

$$p \succeq q$$
 iff $A(p-q) \in C$.

The utility functions that represent \succeq are the vectors u'_i such that $Au'_i = (u_i, 0)$ for i = 1, ..., n. Thus, as was claimed, for any n, one can find a preference relation \succeq such that n utility functions are needed to represent \succeq .¹⁵

3.3. Additive Representation in the Case of Horse Race Lotteries. Although our main focus in this article is on the theory of risk (as opposed to uncertainty), it is nevertheless interesting that the above representation theorems extend readily to an additive representation result in the widely studied setup of Anscombe and Aumann (1963). In this subsection we briefly present this representation that also paves the way toward extending the present decision-making model to scenarios in which the primitives are the (Savagean) acts as opposed to objective probability distributions.

Let *S* be any finite set that is designated as the *state space*. An *act* is thus any map from *S* to *Z*, and a *horse race lottery* is any map from *S* to the set of simple probability measures $\mathbb{P}_s(Z)$. The objects of choice in the Anscombe–Aumann formulation are the members of $\mathbb{H}(Z) := \mathbb{P}_s(Z)^S$ (which we identify with $\mathbb{P}_s(Z)^{|S|}$). The generic horse race lotteries are denoted as *h*, *g*, etc. Clearly, a preference relation \succeq in this model is a preorder defined on $\mathbb{H}(Z)$.

It is readily observed that the definitions of the independence axiom and the generation by a finite procedure extend in the natural way to the Anscombe–Aumann framework. The extension of Theorem 1 to this framework is given next.

¹⁵ This example can also be used to show that a more complete preference relation may have to be represented by a larger set of utility functions. In general, there is no connection between the amount of completeness a relation may have and the number of utilities that are needed to represent this relation.

THEOREM 3. If \succeq is a preference relation on $\mathbb{H}(Z)$ that is generated by a finite procedure, then there exists a set $\mathcal{U} \subseteq \mathbf{C}(Z)^{|S|}$ such that

$$h \succeq g$$
 if and only if $\sum_{s \in S} \int_Z u_s \, dh(s) \ge \sum_{s \in S} \int_Z u_s \, dg(s)$ for all $(u_s)_{s \in S} \in \mathcal{U}$

for each $h, g \in \mathbb{H}(Z)$.

Therefore, finitely generated preferences over horse race lotteries entail a set of preference functionals over acts, and each member of this set is of the form $f \mapsto \sum_{s \in S} u_s(f(s))$, where u_s is a state-dependent utility function. Of course, the situation would be ideal if we could guarantee that each u_s can be written as $u_s = \lambda_s u$ for some (unique) $\lambda_s \ge 0$ and $u \in \mathbb{C}(Z)$. This would entail a (subjective) expected utility representation that is, in a sense, the "right" version of Theorem 1 for the theory of uncertainty. Just as in the case of Anscombe and Aumann (1963), the independence axiom is, however, not powerful enough to yield this extension. The problem is to strengthen the procedure we have been using so far by combining the independence axiom with some other properties and see if one can get a vectorvalued state-independent utility representation for preferences that are generated by such a procedure. This problem stands open for the moment.

4. A GENERAL UNIQUENESS THEOREM

Take any set \mathcal{U} of continuous functions on Z. We say that this set *represents* the preference relation \succeq on $\mathbb{P}(Z)$ whenever (2) holds for each $p, q \in \mathbb{P}(Z)$. Theorems 1 and 2 thus amount to saying that any preference relation that is generated by a finite procedure can be represented by a set of utility functions \mathcal{U} . This way of looking at things clearly requires us to answer the question of what other sets of continuous functions may represent \succeq .

The problem at hand is, then, to identify the set of all transformations on \mathcal{U} that would leave the representation invariant. In the case of the classical expected utility theory (that is, when \mathcal{U} is a singleton), the answer is well known to consist of the set of all positive affine transformations. We claim that this would remain to be the answer in our multiutility context as well, provided that we suitably define what it means to transform a *set of utilities* in a positively affine way. We develop the idea below.

Obviously, the set $\{\lambda u + \theta : \lambda > 0, \theta \in \mathbf{R} \text{ and } u \in \mathcal{U}\}$ represents \succeq whenever \mathcal{U} represents it. Although this is the end of the story in the classical theory, this set does not really correspond to the "set of all positive affine transformations of \mathcal{U} ." Indeed, such a set should intuitively include all functions that can be written as nonnegative finite linear combinations of the members of \mathcal{U} , that is, all functions in cone(\mathcal{U}).¹⁶ Thus, it may be argued that the set we are after is precisely

¹⁶ We define a (*convex*) *cone* in any vector space as any nonempty convex set that is closed under nonnegative scalar multiplication. The *conical hull* of a set S, denoted cone(S), is then defined as the smallest cone in this vector space that contains S. It is easy to see that cone(S) is the intersection of

 $\operatorname{cone}(\mathcal{U}) + \{\theta 1_Z\}_{\theta \in \mathbf{R}}$, where 1_Z is the indicator function of A. Indeed, it is easy to see that this set indeed represents \succeq whenever \mathcal{U} does so. What is more, we can stop at this point if \mathcal{U} is finite. However, allowing for any subset of $\mathbf{C}(Z)$ as a set that may represent a given preference relation, we are forced to enlarge this set to

$$\langle \mathcal{U} \rangle := \operatorname{cl}(\operatorname{cone}(\mathcal{U}) + \{\theta 1_Z\}_{\theta \in \mathbf{R}})$$

Indeed, it is readily observed that if \mathcal{U} represents \succeq , so does $\langle \mathcal{U} \rangle$.¹⁷ It turns out that this is precisely the largest set of all utility functions that represents \succeq .

DEFINITION. Let \mathcal{U} and \mathcal{V} be any nonempty subsets of $\mathbf{C}(Z)$. We say that \mathcal{V} is a *positive affine transformation* of \mathcal{U} if $\mathcal{V} \subseteq \langle \mathcal{U} \rangle$ and $\mathcal{U} \subseteq \langle \mathcal{V} \rangle$.

Observe that $\{v\}$ is a positive affine transformation of $\{u\}$ iff $v = \lambda u + \theta$ for some $\lambda > 0$ and $\theta \in \mathbf{R}$. Therefore, the notion of positive affine transformation of a set of utilities generalizes the standard notion that is defined for a single utility function. What is more, this concept allows us to generalize the uniqueness part of the celebrated expected utility theorem.

THEOREM 4. Let $\mathcal{U} \subseteq \mathbb{C}(Z)$ represent a preference relation \succeq on $\mathbb{P}(Z)$. Then a set \mathcal{V} in $\mathbb{C}(Z)$ represents \succeq if, and only if, it is a positive affine transformation of \mathcal{U} .

In particular, this result shows that \mathcal{U} in Theorem 1 can be chosen to be closed in $\mathbf{C}(Z)$.

5. UTILITY FOR MONEY

In this section, we specialize in the economically interesting case where Z is composed of monetary outcomes. Consequently, we shall consider Z as a compact subset of the reals. Without loss of generality, then, we set Z = [0, 1] in what follows, and view $\mathbb{P}(Z)$ as the set of all lotteries with monetary prizes.

In this specialized setting, preference relations that are generated by a finite procedure are of limited interest. This is because the rationality requirement of positing that the decision maker is able to rank with precision the degenerate lotteries is truly minimal. We shall, therefore, assume here that an agent always strictly prefers δ_z to $\delta_{z'}$ whenever z > z'. Of course, the decision maker might have some other primitive rankings in mind, but she should certainly not have any difficulty in ranking the degenerate lotteries in this way. This intuition, in turn, leads us to call a set R in $\mathbb{P}(Z)^2$ monotonic if

all cones that contain S and hence, that it is composed of all vectors of the form $\sum_{i=1}^{k} \lambda_i x^i$ where $k \in \mathbf{N}$, and $\lambda_i \ge 0$ and $x^i \in S$, i = 1, ..., k.

¹⁷ To see this, take any sequence (u_m) in cone $(\mathcal{U}) + \{\theta 1_Z\}_{\theta \in \mathbf{R}}$ and assume that $||u_m - u||_{\infty} \to 0$. Given any $p, q \in \mathbb{P}(Z)$, if $\int_Z u_m dp \ge \int_Z u_m dq$ for each *m*, then, since the convergence is uniform, we find

$$\int_{Z} u \, d(p-q) = \lim \int_{Z} u_m \, d(p-q) \ge 0$$

This simple observation proves the claim.

$$\{(\delta_z, \delta_{z'}) : z > z'\} \subseteq P_R$$

and suggests the following notion of procedural rationality:

DEFINITION. Let Z = [0, 1]. A preference relation \succeq on $\mathbb{P}(Z)$ is said to be generated by a monotonic procedure if there exists a monotonic set R such that $\succeq = \succeq^R$.

What do monotonically generated preferences look like? Intuitively, one may suspect that they must have something to do with the first-order stochastic dominance. Indeed, Example 1 provides some formal support to this effect. It turns out that we can generalize this example substantially if we restrict our attention to the *continuous* preference relation \succeq on $\mathbb{P}(Z)$ that, by definition of continuity, satisfies the following property: For any sequences p_m and q_m in $\mathbb{P}(Z)$, $p_m \stackrel{w}{\to} p$, $q_m \stackrel{w}{\to} q$, and $p_m \succeq q_m$ imply $p \succeq q$.¹⁸

The following proposition identifies the relation between monotonically generated continuous preferences and the first-order stochastic dominance.

PROPOSITION 1. Let Z = [0, 1]. The first-order stochastic dominance ordering \geq_{FSD} on $\mathbb{P}(Z)$ is the smallest continuous preference relation that is generated by a monotonic procedure. Moreover, a continuous preference relation on $\mathbb{P}(Z)$ is generated by a monotonic procedure if, and only if, it is a proper extension of \geq_{FSD} that satisfies the independence axiom.

It is often argued that consistency with the first-order stochastic dominance is an essential ingredient of rational behavior (cf. Karni and Schmeidler, 1991). Proposition 1 provides an alternative support for this statement by implementing the notion of procedural rationality that we propose in this article. Observe that this result also contributes to the usual characterization of the first-order stochastic dominance (which is sometimes referred to as the majorization theorem in mathematics). Indeed, this theorem entails that $\geq_{\text{FSD}} = \text{cl}(\succeq^R)$ where $R = \{(\delta_z, \delta_{z'}) : z > z'\}$. Given the important role that the first-order stochastic dominance plays in the theory of risk, this observation perhaps deserves explicit mention.

COROLLARY 1. \geq_{FSD} is the smallest continuous preorder that satisfies the independence axiom and properly extends the preorder $\{(\delta_z, \delta_{z'}) : z > z'\}$.

The next step is to introduce the notion of risk aversion to the present setting. Given the procedural decision-making approach that we advance here, the natural way of doing this is to posit that a risk-averse agent is one who ranks a given lottery p below the lottery degenerate at E(p) as a primitive judgement. In fact, it suffices

¹⁸ We could dispense with the continuity assumption by confining attention to simple lotteries; the proof of Proposition 1 demonstrates this formally. At any rate continuity is a mild requirement that is rarely questioned.

for our purposes to postulate that the agent has such an initial ranking in mind only for simple lotteries. Thus, we refer to a set R in $\mathbb{P}(Z)^2$ as *risk averse* if

$$\{(\delta_{E(p)}, p) : p \in \mathbb{P}_s(Z) \text{ and } p \text{ is nondegenerate}\} \subseteq P_R$$

This suggests the following:

DEFINITION. Let Z = [0, 1]. A preference relation \succeq on $\mathbb{P}(Z)$ is said to be generated by a risk-averse procedure if there exists a risk-averse set R such that $\succeq = \succeq^R$.

Given the connection between the first-order stochastic dominance and monotonically generated preferences, the following fact will not come as a surprise.

PROPOSITION 2. Let Z = [0, 1]. A continuous preference relation on $\mathbb{P}(Z)$ is generated by a risk-averse procedure if, and only if, it is a proper extension of the second-order stochastic dominance ordering \geq_{SSD} that satisfies the independence axiom.

Of course, this proposition entails that an exact analogue of Corollary 1 holds in the case of the second-order stochastic dominance. That is, \geq_{SSD} is the smallest continuous preorder that satisfies the independence axiom and properly extends the preorder { $(\delta_{E(p)}, p) : p \in \mathbb{P}_s(Z)$ }.

Since it is standard to model economic agents as money loving and risk averse, it is appealing to combine the monotonic and risk-averse procedures to obtain what might be called the "standard procedure." Formally,

DEFINITION. Let Z = [0, 1]. A preorder \succeq on $\mathbb{P}(Z)$ is said to be generated by a standard procedure if there exists a monotonic and risk-averse set R such that $\succeq = \succeq^R$.

It is conventional to model the preferences of an individual by means of a strictly increasing and concave utility function in applications. The following theorem shows that if the preferences of an agent are generated by a standard procedure, then they can be modeled just as in the conventional theory, but this time using a *set* of increasing and concave utility functions. To state this formally, let us agree to refer to a family \mathcal{U} of real functions on Z as *strictly increasing and concave* whenever (i) all $u \in \mathcal{U}$ is increasing and concave; (ii) for each a < b, we have u(a) < u(b) for some $u \in \mathcal{U}$; and (iii) for each $a \neq b$, we have $\lambda u(a) + (1 - \lambda)u(b) < u(\lambda a + (1 - \lambda)b)$ for some $u \in \mathcal{U}$.

The following is the main result of this section.

THEOREM 5. Let Z = [0, 1]. A continuous preference relation \succeq on $\mathbb{P}(Z)$ is generated by a standard procedure if, and only if, there exists a strictly increasing and concave set \mathcal{U} of continuous utility functions on Z such that

(5)
$$p \succeq q$$
 if and only if $\int_Z u \, dp \ge \int_Z u \, dq$ for all $u \in \mathcal{U}$

for each $p, q \in \mathbb{P}(Z)$.

This result puts us in a position of examining the implications of our formalization of procedural rationality in some economic environments. We shall do this next.

6. ELEMENTARY APPLICATIONS

Although Theorem 5 introduces quite a bit of structure to the analysis, it is not really clear if nontrivial predictions can be obtained on the basis of preferences that are generated, say, by a standard procedure (that may or may not accord with the classical results). If all behavior can be "explained" by such preferences, then the predictive power of the theory would clearly be nil. On the other hand, if we could obtain no clear prediction by this theory, then the situation would again be unsatisfactory. In this section, by using the standard risky choice problems of portfolio selection and insurance demand, we shall demonstrate that the present approach has reasonable scope for applications.

6.1. Preference for Diversification and Procedural Decision Making. We define an asset as a random variable on the measurable space $([0, 1], \mathcal{B}, \ell)$ where \mathcal{B} is the Borel σ -algebra on Z := [0, 1], and ℓ is the Lebesgue measure. The probability measure that an asset *a* induces on [0, 1] is denoted by p_a . Naturally, we derive an individual's preferences over assets from her preference relation \succeq over lotteries, and write $a \succeq^* b$ iff $p_a \succeq p_b$ for any assets *a* and *b*. This definition allows us to talk about preferences for assets that are generated by a certain procedure.

Following Dekel (1989), then, we say that \succeq^* exhibits preference for diversification if for any collection of assets a^1, \ldots, a^m ,

$$a^1 \sim^* \cdots \sim^* a^m$$
 implies $\sum \alpha_i a^i \succeq^* a^1$

for all $(\alpha_i) \in \Delta_{m-1}$ where Δ_{m-1} is the *m*-dimensional simplex. It is well known that an expected utility maximizer (with an increasing von Neumann–Morgenstern utility for money) is risk averse iff she exhibits preference for diversification. This observation is generalized in various directions in the literature. For instance, Dekel (1989) shows that the equivalence continues to hold if the preferences of the individual are represented by any continuous and monotonic (in the sense of \geq_{FSD}) preference functional that is quasiconcave in probabilities. A natural question in our setting then is if the same equivalence obtains in the case of preferences that are generated by a standard procedure. It turns out that the answer is yes, and the potential incompleteness of such preferences does not forbid such a result.

Our first claim is, then, that any continuous preference relation for assets \gtrsim^* that is generated by a standard procedure exhibits preference for diversification. The proof of this claim is essentially identical to that of the corresponding result in the standard theory, but to illustrate that incompleteness never gets in the way, we

quickly go through the argument. Begin by observing that, given such a preference relation \succeq^* , we can use Theorem 5 to find a strictly increasing and concave set \mathcal{U} of continuous utility functions on Z such that

(6)
$$a \succeq^* b \quad \text{iff} \quad \int_Z u \, dp_a \ge \int_Z u \, dp_b \quad \text{for all } u \in \mathcal{U}$$

for any two assets *a* and *b*. Now assume that $a^1 \sim^* \cdots \sim^* a^m$ for any given $m \ge 1$, pick any $u \in U$, and fix any $(\alpha_i) \in \Delta_{m-1}$. We have

$$\int_{Z} u \, dp_{\Sigma \alpha_{i} a^{i}} = \int_{Z} u \bigg(\sum \alpha_{i} a^{i}(\omega) \bigg) d\omega \ge \sum \alpha_{i} \int_{Z} u(a^{i}(\omega)) \, d\omega = \sum \alpha_{i} \int_{Z} u \, dp_{a^{i}}$$

by Jensen's inequality. But (6) guarantees that $\int_Z u \, dp_{a^i} = \int_Z u \, dp_{a^1}$ for each *i* and $u \in \mathcal{U}$, so using the above expression and (6) again, we obtain $\sum \alpha_i a^i \succeq^* a^1$ as is sought.

Interestingly, a partial converse of this observation also holds: If \gtrsim^* is continuous, monotonically generated, and exhibits a preference for diversification, then it must be generated by a standard procedure. This fact is not of immediate interest for our present purposes, and hence, we shall omit its proof. We should note, however, that a proof can be given by modifying the proof of Proposition 2 of Dekel (1989) in a minor way.

In summary, we maintain that the classical result establishing the equivalence of preference for diversification and risk aversion in the case of expected utility maximization carries over to the present framework where the preferences are allowed to be incomplete.

6.2. Actuarial Fairness and Procedural Decision Making. We examine next a textbook example of an insurance model. Consider a decision maker with initial wealth w > 0 and preferences that are generated by a standard procedure. Assume that the agent is facing a risky situation in which she can lose 1 dollar with probability $\pi \in (0, 1)$. The problem of the agent is then to decide how many units $\iota \in [0, 1]$ of insurance to buy given that the cost and the payment (in case of loss) of the unit insurance are c > 0 and 1, respectively. It is well known that if the individual is risk averse, and if insurance is actuarially fair (that is, $c = \pi$), then $\iota = 1$, i.e., the agent insures fully. What is more, this result is proved for an arbitrary concave utility function, and hence Theorem 5 implies readily that it is valid in the case of a continuous preference relation that is generated by a standard procedure.

In fact, such an individual does not even have to use any procedure, she is actually endowed with a primitive judgement that delivers the optimal choice directly. For, the problem of the agent is to find the best lottery in the set $\{p_i\}_{i \in [0,1]}$ where p_i is the lottery that pays $w - \iota c - 1 + \iota$ with probability π and $w - \iota c$ with probability $(1 - \pi)$. So $E(p_i) = w - \iota c$ when $c = \pi$, whereas we have $p_1 = \delta_{w - \iota c}$. Thus, by definition of a risk-averse procedure, it must trivially be the case that p_1 is the unique best choice for any individual with preferences that are generated by a standard procedure.

On the other hand, the classical theory maintains the converse conclusion as well: An expected utility maximizer with strictly increasing and strictly concave utility function underinsures if insurance is *actuarially unfair* (that is, $c > \pi$). Interestingly, this result does not hold for an arbitrary individual with preferences that are generated by a standard procedure. Indeed, if $\succeq \subseteq \mathbb{P}(Z)^2$ is given by (5) with \mathcal{U} being equal to the set of all continuous, strictly increasing, and strictly concave utility functions, then p_1 remains in the \succeq -maximal subset of $\{p_t\}_{t \in [0,1]}$.¹⁹

One may object to this conclusion in that all members of $\{p_i\}_{i \in [0,1]}$ are $\gtrsim \max$ maximal when $c > \pi$. This is, however, not really a cause for concern. All that is to be concluded from this observation is that \succeq is "too incomplete" and that we need to introduce further structure to \mathcal{U} (by strengthening the employed procedure or by other means). Or, alternatively, it may be argued that the choice procedure of the agent should be complemented, say, by introducing a status quo bias to the model. Informally speaking, this would entail that the agent would frequently (but not always) depict a genuine reluctance to change her current insurance policy (because she would do so only if an alternative dominates the status quo at all dimensions, that is, for all u in \mathcal{U}), not a completely uninteresting prediction.

7. FUTURE RESEARCH

In this article, we have proposed a procedural decision-making model, and tried to argue that such models hold realistic promise for future developments of the individual decision theory. We see ourselves not providing a complete theory here, but rather formally suggesting an alternative way of looking at things. The premise of the approach is based on beginning with a core preference relation and then extending that relation by means of a procedure. Although our entire development here assumes a particular procedure, one that uses the independence axiom directly, this is only because we view this simple (but admittedly demanding) procedure as a good starting point. Future thinking about the problem must worry about weakening this procedure. Moreover, extending the present development to the subjective probability framework is left for future research. Finally, the potential incompleteness of procedurally generated preference relations leaves room for introducing status quo bias and other empirically observed behavior traits to decision models. Doing this formally awaits further research as well.

8. PROOFS

The proofs of all of our representation results are based on the following characterization of \succeq^{R} . The basic geometric technique of proof is inspired by the seminal work of Aumann (1962). (See Remark 2 below.)

¹⁹ The original result is recovered only if there is a uniform bound on the extent of concavity of the functions that belong to \mathcal{U} (as will be the case if \mathcal{U} is a closed subset of continuous, strictly increasing and strictly concave real functions on [0, 1]). Actually, this problem is not very different than what is implicitly faced in the traditional approach. It is well known that the more risk averse a given agent (in the sense of Arrow–Pratt), the closer to 1 is the optimal ι .

LEMMA 1. Let R be any subset of $\mathbb{P}(Z)^2$ such that \succeq^R exists. Then, for any $p, q \in \mathbb{P}(Z)$, we have $p \succeq^R q$ if and only if there exists a positive integer k such that

$$p-q = \sum_{i=1}^{k} \lambda_i (p^i - q^i)$$
 for some $\lambda_i \ge 0$ and $(p^i, q^i) \in R, i = 1, ..., k$

PROOF. Take any $p, q \in \mathbb{P}(Z)$ with $p - q = \sum^k \lambda_i (p^i - q^i)$ for some $k \in \mathbb{N}$, $\lambda_i \ge 0$, and $(p^i, q^i) \in R$ for each i = 1, ..., k. Define

(7)
$$\alpha_j := \frac{\lambda_j}{1 + \sum \lambda_i}, \ j = 1, \dots, k \text{ and } \alpha_{k+1} := \frac{1}{1 + \sum \lambda_i}$$

and note that $(\alpha_1, \ldots, \alpha_{k+1}) \in \Delta_k$. Thus, since the preorder \succeq^R satisfies the independence axiom, and $p^i \succeq^R q^i$ for each *i*, we have

$$\alpha_{k+1}p + \sum_{i=1}^{k} \alpha_i q^i = \alpha_{k+1}q + \sum_{i=1}^{k} \alpha_i p^i \succeq^R \alpha_{k+1}q + \sum_{i=1}^{k} \alpha_i q^i$$

By using the independence axiom again, we find then that $p \succeq^{R} q$.

Now let \supseteq stand for the set of all $(p, q) \in \mathbb{P}(Z)^2$ such that $p \sim q = \sum^k \lambda_i (p^i - q^i)$ holds for some $k \in \mathbb{N}$, $\lambda_i \ge 0$, and $(p^i, q^i) \in R$, i = 1, ..., k. By the above argument, we have $\supseteq \subseteq \succeq^R$. To complete the proof, then, it is enough to show that \supseteq is a proper extension of R that satisfies the independence axiom. Whereas reflexivity of \supseteq and that $R \subseteq \supseteq$ are obvious facts, the transitivity and independence of \supseteq follow from routine arguments. To establish the properness of \supseteq , we need to show that $(p,q) \in P_R$ implies the impossibility of $q \supseteq p$. To see this, let $q \supseteq p$ so that $q - p = \sum^k \lambda_i (p^i - q^i)$ where $k \in \mathbb{N}$, $\lambda_i \ge 0$, and $(p^i, q^i) \in R$ for each i = 1, ..., k. Using (7), then, we find $\alpha_{k+1}p + \sum^k \alpha_i p^i = \alpha_{k+1}q + \sum^k \alpha_i q^i$. But if $(p,q) \in P_R$, this is impossible, for we must have $p >^R q$ by properness, and hence from the independence axiom it follows that $\alpha_{k+1}p + \sum^k \alpha_i p^i >^R \alpha_{k+1}q + \sum^k \alpha_i q^i$, which then contradicts the reflexivity of \gtrsim^R .

Among many implications of Lemma 1 is a necessary and sufficient condition (on *R*) for \succeq^R to exist. Although we do not need this characterization in the proof of Theorem 1, we state it now for future reference.

LEMMA 2. Let R be any nonempty subset of $\mathbb{P}(Z)^2$. The preorder \succeq^R exists if and only if

(8)
$$\{q - p : (p, q) \in P_R\} \cap \operatorname{cone}\{p - q : (p, q) \in R\} = \emptyset$$

So, if $P_R = \emptyset$ or $0 \notin co\{p - q : (p, q) \in R\}$, then \succeq^R exists.

PROOF. Suppose that \succeq^R is well defined but (8) fails to hold. Then there exists $(p,q) \in P_R$ such that $q - p = \sum^k \lambda_i (p^i - q^i)$ for some $k \in \mathbb{N}$, $(\lambda_i) \in \mathbb{R}^k_+$, and $((p^i,q^i)) \in \mathbb{R}^k$. Thus, by Lemma 1, $q \succeq^R p$, whereas $p \succ^R q$ must hold due to the properness of \succeq^R , a contradiction. Conversely, assume that (8) holds. We claim that the preorder \succeq defined in the proof of Lemma 1 is actually a proper extension of *R* that satisfies the independence axiom and (1). Given what is shown in the second part of that proof, we only need to establish here the properness of \succeq . To this end, observe that, for any $(p', q') \in P_R$, $q' \succeq p'$ would entail $(q', p') \in$ cone $\{p - q : (p, q) \in R\}$, and this would contradict (8).

To complete our preparation for the proof of Theorem 1, we need to borrow the following result from geometric functional analysis.

LEMMA 3. The conical hull of any finite set in a Hausdorff topological vector space is closed.

PROOF. See Aliprantis and Border (1999, p. 197, Corollary 5.68).

We are now ready to give the proof of Theorem 1.

PROOF OF THEOREM 1. Define the linear space

$$\mathbb{V}(Z) := \operatorname{span}(\mathbb{P}(Z))$$

and make this space a locally convex Hausdorff topological vector space by using the topology of weak convergence. This entails that any net (μ_{α}) in $\mathbb{V}(Z)$ converges to μ , denoted $\mu_{\alpha} \xrightarrow{w} \mu$, if and only if

$$\int_Z f \, d\mu_\alpha \to \int_Z f \, d\mu \quad \text{for all } f \in \mathbf{C}(Z)$$

It is important to note that $\mathbb{P}(Z)$ is a compact and metrizable topological subspace of $\mathbb{V}(Z)$.

We write $R = \{(p^1, q^1), \dots, (p^k, q^k)\}$ and define the set

$$\mathcal{A} := \left\{ \mu \in \mathbb{V}(Z) : \mu = \sum_{i=1}^{k} \lambda_i (p^i - q^i) \text{ for some } \lambda_1, \dots, \lambda_k \ge 0 \right\}$$

that is, $\mathcal{A} = \operatorname{cone}\{p^i - q^i : i = 1, ..., k\}$. \mathcal{A} is easily checked to be convex. By using Lemma 3, therefore, we can conclude that \mathcal{A} is a closed convex cone in $\mathbb{V}(Z)$. So, by the geometric form of the Hahn–Banach theorem, \mathcal{A} must be the intersection of all the closed half spaces, corresponding to hyperplanes through the origin, that contain \mathcal{A} . Of course, this means that there exists a set $\Omega \subseteq \mathbb{V}(Z)^*$ of continuous linear functionals (the members of which correspond to the said set of closed half spaces) such that $\mu \in \mathcal{A}$ if and only if $T(\mu) \ge 0$ for each $T \in \Omega$.

Now fix any $T \in \Omega$, and define $u_T : Z \to \mathbf{R}$ with $u_T(z) := T(\delta_z)$. Since T is continuous on \mathbb{V} , it is obviously continuous on $\{\delta_z : z \in Z\}$, which means that u_T is continuous on Z. We next claim that

(9)
$$T(p) = \int_{Z} u_T dp \text{ for all } p \in \mathbb{P}_s(Z)$$

Observe that this claim holds trivially for any p with a singleton support, and to continue with induction, assume that it is true for all $p \in \mathbb{P}_s(X)$ with |S(p)| = m - 1 where m > 2. Now take any $q \in \mathbb{P}_s(X)$ with |S(q)| = m, and define

$$r := \sum_{z \in S(q) \setminus \{z_0\}} \left(\frac{q(z)}{1 - q(z_0)} \right) \delta_z$$

Clearly, $q = q(z_0)\delta_{z_0} + (1 - q(z_0))r$ so that, by linearity of *T* and the induction hypothesis,

$$T(q) = q(z_0)T(\delta_{z_0}) + (1 - q(z_0))T(r) = q(z_0)T(\delta_{z_0}) + \sum_{z \in S(r)} (1 - q(z_0))r(z)T(\delta_z)$$
$$= \sum_{z \in S(q)} u_T(z)q(z)$$

which proves (9).

Now take any $p \in \mathbb{P}(Z)$ and recall that $\mathbb{P}_s(Z)$ is dense in $\mathbb{P}(Z)$ with respect to the topology of weak convergence. Therefore, given that $\mathbb{P}(Z)$ is metrizable in this topology, we can find a sequence (p_m) in $\mathbb{P}_s(Z)$ with $p_m \xrightarrow{w} p$. Since *T* is continuous on $\mathbb{P}(Z)$, we have $T(p_m) \to T(p)$. On the other hand, $p_m \xrightarrow{w} p$ implies that $\int_Z u_T dp_m \to \int_Z u_T dp$ since $u_T \in \mathbf{C}(Z)$. Therefore, by (9), we find $T(p_m) \to \int_Z u_T dp$, allowing us to improve (9) to

(10)
$$T(p) = \int_{Z} u_T \, dp \quad \text{for all } p \in \mathbb{P}(Z)$$

Finally, take any $p, q \in \mathbb{P}(Z)$, and note that $p \succeq q$ iff $p - q \in A$ by Lemma 1. But by definition of Ω we have $p - q \in A$ iff $T(p - q) \ge 0$ for all $T \in \Omega$, which holds iff $T(p) - T(q) \ge 0$ for all $T \in \Omega$ due to the linearity of *T*s. Using (10), therefore, we have

$$p \succeq q$$
 iff $\int_Z u \, dp \ge \int_Z u \, dq$ for all $u \in \mathcal{U}$

where \mathcal{U} stands for the convex hull of $\{u_T : T \in \Omega\}$. The proof is complete.

REMARK 2. Implicit in this proof is that \succeq^R admits a representation of the form (2) whenever it is well defined and cone{ $p - q : (p, q) \in R$ } is closed in the linear span of $\mathbb{P}(Z)$. This observation, along with Lemma 1, yields the general strategy

of proof behind our representation theorems. As noted earlier, this strategy is suggested by the work of Aumann (1962).

PROOF OF THEOREM 2. Let $\succeq = \succeq^R$ where $R = \{(p^1, q^1), \dots, (p^k, q^k)\} \subseteq \mathbb{P}(Z)$. Define

$$\mathcal{A} := \left\{ x \in \mathbf{R}^{|Z|} : x = \sum_{i=1}^{k} \lambda_i (p^i - q^i) \text{ for some } \lambda_1, \dots, \lambda_k \ge 0 \right\}$$

This set is obviously the smallest convex cone that is generated by a finite number of points, and thus by the Minkowski–Weyl theorem, it must be polyhedral.²⁰ This means that there exist finitely many functions $u_1, \ldots, u_n \in \mathbf{R}^Z$ such that $\mathcal{A} = \{x \in \mathbf{R}^{|Z|} : \sum_{z \in Z} u_j(z) x(z) \ge 0$ for each $j\}$. But by Lemma 1, we have $p \succeq q$ iff $p - q \in \mathcal{A}$ for each $p, q \in \mathbb{P}(Z)$, and combining these two findings establishes (3).

Conversely, take any preorder \succeq that is defined on $\mathbb{P}(Z)$ by (3) for some $\mathbf{u}: Z \to \mathbf{R}^n$, $n \in \mathbf{N}$. Define $\mathcal{B} := \bigcup_{\lambda > 0} \lambda(\mathbb{P}(Z) - \mathbb{P}(Z))$ and note that $\{x \in \mathcal{B} : \sum_{z \in Z} u_j(z)x(z) = 0\}$ is a hyperplane in $\mathbf{R}^{|Z|}$ that passes through the origin. Next define $\mathcal{A} := \bigcap_{j=1}^n \{x \in \mathcal{B} : \sum_{z \in Z} u_j(z)x(z) \ge 0\}$ and note that $p \succeq q$ iff $p - q \in \mathcal{A}$ for any $p, q \in \mathbb{P}(Z)$. Since \mathcal{A} is a polyhedral cone, it must be finitely generated by the Minkowski–Weyl theorem. This means that there exist finitely many x^1, \ldots, x^k in \mathcal{B} such that $\mathcal{A} = \{\sum_{i=1}^k \lambda_i x^i : \lambda_1, \ldots, \lambda_k \ge 0\}$. Clearly, we can assume that each x^i can be written as the difference of two probability vectors p^i and q^i . Now define

$$R := \{ (p^i, q^i) : i = 1, \dots, k \} \cup \{ (q^i, p^i) : q^i - p^i \in \mathcal{A} \}$$

Clearly, if $(p^i, q^i) \in P_R$, then $q^i - p^i \notin A$ so that $p^i \succ q^i$ must hold. Thus \succeq is a proper extension of *R* that satisfies the independence axiom, which, in turn, implies that \succeq^R exists. But then, by Lemma 1, $p \succeq^R q$ iff $p - q \in A$, and hence, $\succeq = \succeq^R$.

PROOF OF THEOREM 3. Since the proof follows the same chain of reasoning that yielded Theorem 1, it will suffice here to provide only a sketch. We now take $\mathbb{V}(Z)$ as the span of $\mathbb{H}(Z)$, and define $\mathcal{A} := \operatorname{cone}\{h - g : (h, g) \in R\}$. By Lemma 3, \mathcal{A} is a closed convex cone, and hence we may use the Hahn–Banach theorem to find a set $\Omega \subseteq \mathbb{C}(\mathbb{V}(Z))$ such that $\mu \in \mathcal{A}$ iff $T(\mu) \ge 0$ for all $T \in \Omega$. By the linearity of any such T, we have, for all $\mu = (\mu_s)_{s \in S} \in \mathbb{V}(Z)$,

$$T((\mu_s)_{s\in S}) = T\left(\sum_{s\in S} (\mu_s, 0_{-s})\right) = \sum_{s\in S} T(\mu_s, 0_{-s}) = \sum_{s\in S} T_s(\mu_s)$$

²⁰ The version of the Minkowski–Weyl theorem we use here states that a convex cone in \mathbf{R}^d is polyhedral (i.e., it is the intersection of a finite number of closed half spaces) if, and only if, it is the smallest convex cone that is generated by a finite number of points (i.e., it is a finitely generated cone) in \mathbf{R}^d . See, for instance, Rockafellar (1970, Theorem 19.1).

where $T_s(\sigma) := T(\sigma, 0_{-s})$ for each $\sigma \in \text{span}(\mathbb{P}_s(Z))$ and $s \in S$. It is readily checked that T_s is linear for each s. Therefore, the standard induction argument (that we have detailed above) yields

$$T_s(p) = \int_Z u_{T,s} \, dp \quad \text{for all } p \in \mathbb{P}_s(Z)$$

for some $u_{T,s} \in \mathbf{C}(Z)$, $s \in S$. But then, after checking that Lemma 1 applies to horse race lotteries without modification, we conclude that (5) holds for each $h, g \in \mathbb{H}(Z)$, provided that we choose $\mathcal{U} := \{(u_{T,s})_{s \in S} : T \in \Omega\}$.

PROOF OF THEOREM 4. Since the "if" part of the claim can be demonstrated quite easily, we shall focus here rather on the "only if" part. Let \mathcal{U} and \mathcal{V} be any two sets that represent \succeq . Our aim is to show that $\langle \mathcal{U} \rangle = \langle \mathcal{V} \rangle$. To derive a contradiction, suppose that we can find a v in $\langle \mathcal{V} \rangle \setminus \langle \mathcal{U} \rangle$. Since $\langle \mathcal{U} \rangle$ is a closed convex set in the Banach space $\mathbb{C}(Z)$, we can strictly separate v from $\langle \mathcal{U} \rangle$ by a closed hyperplane. That is, there exists a scalar α and a continuous functional $T \in \mathbb{C}(Z)^*$ such that $T(v) > \alpha \ge \sup_{u \in \langle \mathcal{U} \rangle} T(u)$. Observe that we must have $\alpha \ge 0$ since $0 \in \langle \mathcal{U} \rangle$. Moreover, we have

$$\frac{\alpha}{2^m} \ge \frac{1}{2^m} \sup_{u \in \langle \mathcal{U} \rangle} T(u) = \sup_{u \in \langle \mathcal{U} \rangle} T\left(\frac{u}{2^m}\right) = \sup_{u \in \langle \mathcal{U} \rangle} T(u) \quad \text{for all } m = 1, 2, \dots$$

Combining these two observations, it follows that

(11)
$$T(v) > 0 \ge T(u) \text{ for all } u \in \langle \mathcal{U} \rangle$$

By the Riesz representation theorem, on the other hand, there exists a finite signed Borel measure μ on Z such that

(12)
$$T(w) = \int_{Z} w \, d\mu \quad \text{for all } w \in \mathbf{C}(Z)$$

By (11), we have $\alpha \ge \theta T(1_Z) = \theta \mu(Z)$ for any real θ , and this implies that $\mu(Z) = 0$. Clearly, we can write μ as the difference of two positive finite Borel measures on $\mathbb{C}(Z)$, say μ^+ and μ^- . We have $\mu = \mu^+ - \mu^-$ and hence $\mu^+(Z) = \mu^-(Z) = c \ge 0$. If c = 0 was the case, then we would have $\mu^+ = \mu^- = \mu = 0$, which would contradict the fact that T(v) > 0. Thus c > 0, and we can define $p := \mu^+/c$ and $q := \mu^-/c$ as two Borel probability measures on Z. We now invoke (12) to get $T(w)/c = \int_Z w \, dp - \int_Z w \, dq$ for all $w \in \mathbb{C}(Z)$, so that, by (11), we may conclude that

$$\int_{Z} v \, dp > \int_{Z} v \, dq \quad \text{and} \quad \int_{Z} u \, dq \ge \int_{Z} u \, dp \quad \text{for all } u \in \langle \mathcal{U} \rangle$$

But this is a contradiction since these inequalities cannot hold simultaneously given that both \mathcal{U} and \mathcal{V} represent \succeq . We may thus conclude that $\langle \mathcal{V} \rangle \subseteq \langle \mathcal{U} \rangle$. The

converse containment is proved by interchanging the roles of \mathcal{U} and \mathcal{V} in the above argument.

PROOF OF PROPOSITION 1. Define $R = \{(\delta_z, \delta_{z'}) : z > z'\}$, and observe that \succeq^R is well defined by Lemma 2. To prove the first part of Proposition 1, it will thus suffice here to show that \geq_{FSD} equals the closure of \succeq^R . To see this, begin by observing that \geq_{FSD} is a proper extension of *R* that satisfies the independence axiom. Since \geq_{FSD} is continuous, we then have $\text{cl}(\succeq^R) \subseteq \geq_{\text{FSD}}$.

We shall next show that $cl(\succeq^R) \supseteq \ge_{FSD}$. To this end, pick any two simple lotteries $p, q \in \mathbb{P}_s(Z)$ with $p \ge_{FSD} q$. By definition, we have $\sum^i p(z_j) \le \sum^i q(z_j)$ for all i = 1, ..., k, where $S(p - q) = \{z_1, ..., z_k\}$ (for some k) with $z_1 < \cdots < z_k$. Define

$$\lambda_i := \sum_{j=1}^{i} (q(z_j) - p(z_j)), \quad i = 1, \dots, k-1$$

and observe that we then have

 $(p-q)(z_1) = -\lambda_1$ and $(p-q)(z_j) = (\lambda_{j-1} - \lambda_j), \quad j = 2, ..., k$

Combining these equations with the hypothesis that $S(p-q) = \{z_1, ..., z_k\}$, we may write

$$p-q=\sum_{i=1}^{k-1}\lambda_i(\delta_{z_{i+1}}-\delta_{z_i})$$

By applying Lemma 1, therefore, we find $p \succeq^R q$.

To extend this observation to the case of all Borel probability measures, take any $p, q \in \mathbb{P}(Z)$ with $p \geq_{\text{FSD}} q$. It is well known that one can approximate a monotonic function (which is necessarily of bounded variation) by simple functions both from above and below. Applying this fact to the distribution functions induced by p and q, we can find two sequences of simple probability measures (p_m) and (q_m) such that $p_m \stackrel{w}{\to} p$ and $q_m \stackrel{w}{\to} q$ with

$$p_m \geq_{\text{FSD}} p \geq_{\text{FSD}} q \geq_{\text{FSD}} q_m \quad m = 1, 2, \dots$$

Now fix any positive integer M and observe that $p_M \ge_{\text{FSD}} q_m$ for each m. By using the observation established in the previous paragraph, then, we have $p_M \succeq^R q_m$ for each m. This clearly implies that $(p_M, q) \in \text{cl}(\succeq^R)$. Since this is valid for an arbitrary $M \in \mathbb{N}$, passing to the limit one more time yields $(p, q) \in \text{cl}(\succeq^R)$. Hence $\ge_{\text{FSD}} \subseteq \text{cl}(\succeq^R)$ and we are done.

The "if" part of the second part of the proposition is easy: If \succeq is a proper extension of \geq_{FSD} , then $R := \succeq$ is monotonic, and $\succeq = \succeq^R$. To go the other way, we fix a continuous preference relation \succeq on $\mathbb{P}(Z)$ that is generated by a monotonic procedure, and establish two elementary observations.

NOTATION. For any $p \in \mathbb{P}(Z)$, we define $m(p) := \max S(p)$,²¹ and let F_p stand for the distribution function induced by p.

CLAIM 1. $\delta_{m(p)} \succ p$ holds for all $p \in \mathbb{P}(Z)$ with $|S(p)| \ge 2$.

PROOF OF CLAIM 1. $|S(p)| \ge 2$ implies that there exists a $z \in S(p) \setminus \{m(p)\}$ such that F(z) > 0. Pick any $\varepsilon \in (0, F(z))$ and define $p' := \varepsilon \delta_z + (1 - \varepsilon) \delta_{m(p)}$. Since $p' >_{\text{FSD}} p$, we have $p' \succeq p$ by the first part of Proposition 1. But since \succeq is generated by a monotonic procedure, we also have $\delta_{m(p)} \succ p'$, so by transitivity, $\delta_{m(p)} \succ p$.

CLAIM 2. For any $r, q \in \mathbb{P}(Z)$ with $r >_{FSD} q$, if there exist $0 \le a < b \le 1$ with

$$F_r(t) = F_q(t)$$
 for all $t \notin [a, b)$ and $F_r(t) = F_r(a)$ for all $t \in [a, b)$

then $r \succ q$.

PROOF OF CLAIM 2. Since $r >_{\text{FSD}} q$, we have $F_r \leq F_q$ with strict inequality at some point in [0, 1]. So since $F_r = F_q$ on $\mathbb{R} \setminus [a, b)$, we must have $F_r(b) - F_r(a-) = F_q(b) - F_q(a-) =: \beta > 0$. Define the distribution functions

$$G(t) := \begin{cases} \frac{F_r(t)}{1-\beta}, & t < b\\ \frac{F_r(t)-\beta}{1-\beta}, & t \ge b \end{cases} \quad \text{and} \quad H(t) := \begin{cases} \frac{F_q(t)-F_r(t)}{\beta}, & t < b\\ 1, & t \ge b \end{cases}$$

and denote the induced Lebesgue–Stieltjes probability measures by r_G and r_H . It is easily checked that $r = \beta \delta_b + (1 - \beta)r_G$ and $q = \beta r_H + (1 - \beta)r_G$. If r_H is a degenerate lottery, then r > q obtains directly from the definition of monotonic generation. If, on the other hand, $|S(r_H)| \ge 2$, then we apply Claim 1 and the independence axiom to get the desired conclusion.

We are now ready to complete the proof of Proposition 1. Pick any lotteries p and q in $\mathbb{P}(Z)$ with $p >_{FSD} q$. We wish to show that p > q. By definition, we have $F_p \leq F_q$ and $F_p(b) < F_q(b)$ for some $b \in (0, 1)$. Since the continuity points of F_q is dense in **R**, we may assume that b is a continuity point of F_q . Now define

$$a := \inf \left\{ t \in \mathbf{R} : F_q(t) \ge \frac{1}{2} F_p(b) + \frac{1}{2} F_q(b) \right\}$$

Since $F_p(0-) = F_q(0-) = 0$, $F_p(1) = F_q(1) = 1$, and F_q is continuous at b, we have $0 \le a < b \le 1$. We next define

$$R(t) := \begin{cases} \frac{1}{2}F_p(b) + \frac{1}{2}F_q(b), & a \le t < b \\ F_q(t), & \text{otherwise} \end{cases}$$

²¹ Since S(p) is a closed set by definition, and Z is compact, S(p) is compact and hence m(p) is well defined.

and observe that *R* is a distribution function on the line. (The only nontrivial part of this fact is that $R(a-) \leq R(a)$, which follows from the definition of *a*.) Let *r* stand for the Lebesgue–Stieltjes measure induced by *R*. Since, $F_p \leq R$, we have $p \geq_{\text{FSD}} r$, so by the first part of Proposition 1, $p \succeq r$. But Claim 2 applies to *r* and *q*, so we have $r \succ q$, and by transitivity of \succeq , we find $p \succ q$.

PROOF OF PROPOSITION 2. Take *R* to be the set of all $(\delta_{E(p)}, p)$ such that *p* is a simple and nondegenerate probability measure on *Z*. It is readily observed that $\succeq^R \subseteq \geq_{\text{SSD}}$ and hence $\operatorname{cl}(\succeq^R)$ is well defined. The proof will then be complete if we can show that $\operatorname{cl}(\succeq^R) \supseteq \geq_{\text{SSD}}$. We shall do this only in terms of the simple probability measures, for the extension to the general case can be achieved just as in the proof of Proposition 1 by using the denseness of $\mathbb{P}_s(Z)$ in $\mathbb{P}(Z)$.

Take any $p, q \in \mathbb{P}_s(Z)$ such that $p \ge_{SSD} q$, and write $S(p) \cup S(q) = \{z_1, \ldots, z_m\}$. This means that q is a mean preserving spread of p so that there must exist simple probability measures r^1, \ldots, r^m such that

$$q(z_j) = \sum_{i=1}^m p(z_i)r^i(z_j) \quad j = 1, \dots, m$$

and

$$E(r^{i}) = \sum_{j=1}^{m} z_{j}r^{i}(z_{j}) = z_{i}, \quad i = 1, ..., m$$

But then, for each j,

$$q(z_j) + \sum_{i=1}^{m} p(z_i)(\delta_{z_i}(z_j) - r^i(z_j)) = \sum_{i=1}^{m} (p(z_i)r^i(z_j) + p(z_i)(\delta_{z_i}(z_j) - r^i(z_j)))$$
$$= \sum_{i=1}^{m} p(z_i)\delta_{z_i}(z_j)$$
$$= p(z_j)$$

so that, by letting $\lambda_i := p(z_i)$, i = 1, ..., m, we obtain $p - q = \sum^m \lambda_i (\delta_{z_i} - r^i)$, which yields $p \succeq^R q$ by Lemma 1.

The proof of the second part of the proposition is similar to the proof we gave for the corresponding part of Proposition 1, and is omitted here for brevity.

PROOF OF THEOREM 5. Let \succeq be a continuous preference relation on $\mathbb{P}(Z)$ that is generated by a standard procedure so that $\succeq = \succeq^R$ for some monotonic and risk-averse *R*. By Proposition 1, \succeq^R must be a proper extension of the first-order stochastic dominance ordering. Therefore, \succeq^R satisfies all three axioms of Mitra and Ok (2000), and thus applying Theorem 1 of that article, we find a set \mathcal{U} of increasing utility functions such that $p \succeq q$ if and only if $\int_Z u \, dp \ge \int_Z u \, dq$ for all $u \in \mathcal{U}$. That this set must be strictly increasing and concave is easily verified. To prove the converse, assume that there exists a set \mathcal{U} with the stated properties, and let $R := \succeq$. Then R is a monotonic and risk-averse set, and we trivially have $\succeq = \succeq^R$.

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